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T_i-Rough Sets in Tri-topological Spaces

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Abstract:

The primary goal of this paper is to introduce certain new types of rough sets based on approaches that leverage three distinct topologies, collectively known as tri-topological spaces.

This paper presents and analyzes various properties of these newly proposed rough sets by introducing new types of open sets, through which we introduce the concepts of T_i −iinterior and T_i −closure operators of any non-empty set, which express the T_i −lower and T_i − upper approximations to it. Finally, the paper explores the relationships between these types, highlighting their main properties within tri-topological spaces, supported by relevant theories and illustrative examples.

Keywords: Tri-topological spaces, T_i-open sets, T_i −iinterior and T_i −closure, T_i −lower and T_i − upper approximations and T_i-rough sets, $for\ i = 1,2,3,4$.

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المجموعات التقريبية في الفضاءات الثلاثية التبولوجية من النوع T_i

انتصار الأمين

قسم الرياضيات –كلية العلوم–جامعة الزاوية

الملخص:

الهدف الأساسي من هذه الورقة هو تقديم أنواع جديدة من المجموعات التقريبية بناءًا على فضاءات جديدة مكونة من ثلاث طبولوجيات مختلفة معرفة على نفس المجموعة، تُعرف مجتمعة باسم الفضاءات الطوبولوجية الثلاثية.

يعرض هذا البحث ويحلل الخصائص المختلفة لهذه المجموعات التقريبية المقترحة حديثًا من خلال تقديم أنواع جديدة من المجموعات المفتوحة، والتي من خلالها نقدم مفهومي الداخلية والغلاقة الجديدين واللذان يعبران عن مفهومي التقريبات السفلية والعلوية لتلك المجموعة، كذلك يوضح انه بالإمكان ان نفس المجموعة يمكن أن تكون إما تقريبية أو معرفة، اعتمادًا على المعايير التي نقيسها بها، مما يساعد على إصدار أحكام أكثر دقة في المواقف المعقدة المختلفة. وأخيراً، يستكشف البحث العلاقات بين هذه الأنواع، ويسلط الضوء على خصائصها الرئيسية ضمن الفضاءات الطوبولوجية الثلاثية، مدعماً بالنظريات ذات الصلة والأمثلة التوضيحية.

الكلمات المفتاحية: الفضاءات الثلاثية التبولوجية، المجموعات المفتوحة من النوع T_i، الداخلية من النوع T_i، الغلاقة من النوع T_i، التقريب السفلي من النوع T_i، التقريب العلوي من النوع T_i ، المجموعات التقريبية من النوعT_i حيث $i = 1,2,3,4$

1. **Introduction**

The concept of rough sets was initially introduced by Zdzisław Pawlak [1] in 1982 as a mathematical tool to handle vagueness and uncertainty in data analysis under equivalence relation. Rough sets provide a framework for approximating a set

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by two crisp sets, known as the lower and upper approximations, which enclose the uncertain set. While traditional rough sets are defined within the context of a single topology, the complexity of real-world data often requires more sophisticated structures to capture the nuances of uncertainty. In [2] Abu-Donia introduced multi knowledge bases using rough approximations and topology. He with Salama in [3, 4] have generalized the classical rough approximation spaces using topological near open sets called δβ-open sets.

In mathematics, the topological structures is one of the most important and widely used ideas. Due to its importance in many applications in most real-life situations, they have begun to develop starting from single topology, it extends to bi topological spaces which introduced by Kelly [5]. A. E.A. Marei [6] studied rough sets on a bi topological view. After that the extension to three-topological spaces was launched for the first time by Martin M. kovar [7] in 2000, where a non-empty set X with three topologies is called tri-topological spaces, a large number of papers have been produced in order to generalize the topological concepts to tri-topological spaces. Also in 2003 Luay. A. [8], has been initiated the systematic study of tri-topological spaces and dealt with in detail and clear. Where they define it as a spaces equipped with three topologies, i.e. triple of topologies on the same set. In 2004 Hassan. A. F. [9] has been studied * δ - open set in tri-topological spaces. Palaniammal. S**.** [10] studied tri-topological spaces. Tapi. U.D., Sharma. R. and Deole. B. [11] introduced semi open set and pre-open set in tri topological space.

Due to the dependence of topological concepts on the definition of the open set, we present new types of open (closed) sets in tri topological spaces and study the relationship between them, and through them we introduce new types of rough sets and their properties. In the second section, some preliminary concepts about tri topological spaces were presented. The main goal of the third section of the manuscript is to present some new types of tri open and tri closed sets in tri topological space with some examples and

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theories. The fourth section aims to define T_i −iinterior and T_i −closure operators. Operations on T_i-tri open (closed) sets are studied in Section 5. Finally, in Section 6, we present concepts of T_i-rough sets and discuss some properties.

2. Tri topological space:

In this section, we introduce some definitions, which is necessary for this paper.

Definition 2.1: [7] Let X be a non empty set and τ_1, τ_2 and τ_3 be three topologies on X. The set X with three topologies is called a Tri topological space. It is denoted by $(X, \tau_1, \tau_2, \tau_3)$.

Definition 2.2: [7] Let $(X, \tau_1, \tau_2, \tau_3)$ be a tri topological space. $A \subseteq X$ is called open set if $A \in \tau_1 \cup \tau_2 \cup \tau_3$.

The complement of A is called closed set.

Example 2.1: Let X = {a, b, c} τ_1 ={ φ, X}, τ_2 ={$\varphi, \{a\}, X$}, τ_3={$\varphi, \{b\}, X$}. Then $(X, \tau_1, \tau_2, \tau_3)$ is a Tri topological space. The set $A = \{a\}$ is an open set and A^C = {b, c} is closed set in $(X, \tau_1, \tau_2, \tau_3)$

Proposition 2.1: Any topological space is a tri topological space.

Proof: Let (X, τ) be a topological space, then (X, τ, τ, τ) is a Tri topological space.

The opposite of proposition 2.1 is not true in general, as shown in the following example

Example 2.2: In the example 2.1, the space $(X, \tau_1, \tau_2, \tau_3) = \{\varphi, \{a\}, \{b\}, X\}$ is not topological space.

We can induces a topological space from tri topological space in many ways.

Proposition 2.2: Let $(X, \tau_1, \tau_2, \tau_3)$ be a tri topological space. Then

1. The set X with intersection of all topologies is a topological space and denoted by τ_I , i.e. $\tau_I = \tau_1 \cap \tau_2 \cap \tau_3$.

2. The set X with the supremum and denoted by τ_S, i.e. $\tau_S = \tau_1 \vee \tau_2 \vee \tau_3$ is a topological space.

3. (X, τ_i) are a topological space for all $i = 1,2,3$.

Proof: it is obvious.

Remark 2.1: From Proposition 2.2, the obtained topology in the first part $\tau_I = \tau_1 \cap \tau_2 \cap \tau_3$ is called the induced topology and (X, τ_I) is induced topological space, while in the second part $\tau_S = \tau_1 \vee \tau_2 \vee \tau_3$ is called the supremum topology and (X, τ_S) is supremum topological space contains τ_1, τ_2, τ_3.

Example 2.3: Let $X = \{1,2,3,4\}$, $\tau_1 = \{X, \varphi, \{1\}, \{2\}, \{1,2\}\}$, $\tau_2 = \{X, \varphi, \{1\}, \{3\}, \{1,3\}\}$, $\tau_3 = \{X, \varphi, \{1\}, \{4\}, \{1,4\}\}$. So (X, τ_i) are topological spaces for all $i = 1,2,3$. Then:

The induced topological space (X, τ_I) is $\{X, \varphi, \{1\}\}$.

The supremum topological space (X, τ_S) is $\{\varphi, X, \{1\}, \{2\}, \{1,2\}, \{3\}, \{1,3\}, \{4\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \{1,2,3\}, \{1,2,4\}, \{1,3,4\}, \{2,$

We will examine some possible possibilities for topological stacks, and test which ones constitute a topology in and of themselves and which ones do not. Accordingly, we introduce new types of openness in this tri topological space, as we will see in the following paragraphs.

3. New Types of Openness in Tri topological Space:

In this section, we will discuss some possible definitions for the different types of tri open sets and their properties with some illustrative examples.

Definition 3.1: Let $(X, \tau_1, \tau_2, \tau_3)$ be a tri topological space. Let $A \subseteq X$. We distinguish four cases:

1. If A is open in all topologies (i.e) if it satisfying the condition $A \in \tau_I$, where $\tau_I = \tau_1 \cap \tau_2 \cap \tau_3$. Then A is called a tri open set of Type 1 in X and denoted to it as (T_1-tri open set). In other words, A is T_1-tri open set in X if A is open in the induced topology τ_I. The complement of A is called T_1-tri closed set in tri topological space.

2. If A is open in only one of the topological spaces (X, τ_i) for $i = 1,2,3$. (i,e) $A \in \sigma_O$, where $\sigma_O =$ only one of open sets of $\tau_i, i = 1,2,3$. Then A is called a tri open set of Type 2 in X and we denoted to it as (T_2-tri open set). The complement of A is called T_2-tri closed set in tri topological space.

3. If A is open in any one of the topological spaces (X,τ_i) for $i = 1,2,3$. (i,e) $A \in \sigma_U, where\ \sigma_U = \tau_1 \cup \tau_2 \cup \tau_3$. Then A is called a tri open set of Type 3 in X and we denoted to it as (T_3-tri open set). The complement of A is called T_3-tri closed set in tri topological space.

4. If A is open in the supremum topology. (i,e) $A \in \tau_S$, where $\tau_S = \tau_1 \vee \tau_2 \vee \tau_3$. Then A is called a tri open set of Type 4 in X and we denoted to it as (T_4-tri open set). The complement of A is called T_4-tri closed set in tri topological space.

Example 3.1: In example 2.3:
T_1-tri -open sets in τ_I are$\{\varphi, \{1\}, X\}$, T_1-tri closed sets are $\{\varphi, X, \{2,3,4\}\}$.
T_2-tri -open sets in σ_O are $\{\{2\},\{1,2\},\{3\},\{1,3\},\{4\},\{1,4\}\}$.
T_2-tri closed sets are$\{\{1,3,4\},\{3,4\},\{1,2,4\},\{2,4\},\{1,2,3\},\{2,3\}\}$.
T_3-tri open sets in σ_U are $\{\varphi, X, \{1\},\{2\},\{1,2\},\{3\},\{1,3\},\{4\},\{1,4\}\}$.
T_3-tri closed sets are $\{X,\varphi,\{2,3,4\},\{1,3,4\},\{3,4\},\{1,2,4\},\{2,4\},\{1,2,3\},\{2,3\}\}$.
T_4-tri open sets in τ_S are $\{\varphi, X, \{1\},\{2\},\{1,2\},\{3\},\{1,3\},\{4\},\{1,4\},\{2,3\},$
$\{2,4\},\{3,4\},\{1,2,3\}, \{1,2,4\},\{1,3,4\},\{2,3,4\}\}$.
T_4-tri closed sets are
$\{X,\varphi,\{2,3,4\},\{1,3,4\},\{3,4\},\{1,2,4\},\{2,4\}\{1,2,3\},\{2,3\},\{1,4\},$
$\{1,3\},\{1,2\},$
$\{4\},\{3\},\{2\},\{1\}\}$.

Remark 3.1: The collections τ_I , τ_S are all topological spaces, while $\sigma_U = \tau_1 \cup \tau_2 \cup \tau_3$ and $\sigma_O = only\ one\ of\ open\ sets\ of\ \tau_i, i = 1,2,3$ are not constitute topological spaces. Note that, for example $\{1,2\},\{3\} \in \sigma_U\ but\ \{1,2\} \cup \{3\} = \{1,2,3\} \notin \sigma_U$. Also, $\{2\},\{3\} \in \sigma_O\ but\ \{2\} \cup \{3\} = \{2,3\} \notin \sigma_O$.

Proposition 3.1: Let (X,τ_I) be induced topological space, we have:

1. φ and X are always T_1-tri open and T_1- tri closed sets.
2. A is T_1-tri open iff A is open with respect to each topology.

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3. A is T_1-tri closed iff A is closed with respect to each topology.
4. A is T_1-tri closed iff A^C is T_1-tri open.

Proof: it is obvious.

Proposition 3.2: Let (X, τ_S) be supermum topological space, we have:

1. φ and X are always T_4-tri open and T_4-tri closed sets.
2. A is T_4-tri open iff A is open with respect to at least one of the three topologies.
3. A is T_4-tri closed iff A is closed with respect to at least one of the three topologies.
4. 4-A is T_4-tri closed iff A^C is T_4-tri open.

Proof: it is obvious.

We can reformulate the previous two propositions in general as in the following theorem

Theorem 3.1: Let $(X, \tau_1, \tau_2, \tau_3)$be a tri topological space. Let $A \subseteq X$.

1. A is T_1-tri open iff A is open with respect to each topology.
2. A is T_1-tri closed iff A is closed with respect to each topology.
3. A is T_1-tri closed iff A^C is T_1-tri open.
4. A is T_4tri open iff A is open with respect to at least one of the three topologies.
5. A isT_4- tri closed iff A is closed with respect to at least one of the three topologies.
6. A is T_4-tri closed iff A^C isT_4- tri open. A is T_1-tri closed iff A^C is T_1-tri open

Proof: it's obvious.

Theorem 3.2: Let $(X, \tau_1, \tau_2, \tau_3)$ be a tri topological space, $A \subseteq X$.

1. If A is an open set in one of the topologies τ_1, τ_2, τ_3 then A is T_3-tri open or T_4-tri open set
2. If A is closed set in one of the topologies τ_1, τ_2, τ_3 then A is T_3-tri closed or T_4-tri closed set.
3. If A is T_2-tri open set then A is an open set in tri topological space.

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4. If A is T_2-tri closed set then A is closed set in tri topological space.

Proof: 1-Let A be an open set in any of the topological spaces (X, τ_i) for$i = 1,2,3$, which means that A is an open set in a supermom topological space, we prove that A is T_3-tri open set since $A \in \tau_1 \cup \tau_2 \cup \tau_3$ which means that A is an open set in one of the topologies τ_1, τ_2, τ_3, but $\tau_1 \cup \tau_2 \cup \tau_3 \subseteq \tau_1 \vee \tau_2 \vee \tau_3$.So $A \in \tau_1 \vee \tau_2 \vee \tau_3$and A is T_3-tri open set.
2- it is obvious
3-it is obvious
4-Let $\{\{B_i, i \in I\}$ be a family of T_1-tri closed sets in X. Let $A_i = {B_i}^C$, so
$\{\{A_i, i \in I\}$ is a family of T_1-tri open sets in X.and since Arbitrary union of T_1-tri open sets is T_1-tri open. Hence $\cup A_i$ is T_1-tri open and so $(\cup A_i)^C$ is T_1-tri closed. i.e., $\cap {A_i}^C$ is T_1-tri closed (i.e.) $\cap B_i$ is T_1-tri closed. Hence, arbitrary intersection of T_1-tri closed sets is T_1-tri closed.

Remark 3.2: The reverse of the previous theorem is not valid as shown in the next example.

Example 3.2: Let $X = \{1,2,3\}, \tau_1 = \{\varphi, \{1\}, X\}$, $\tau_2 = \tau_3 = \{\varphi, \{2\}, X\}$. (X, τ_i) are topological space for all $i = 1,2$. The set $\{1,2\}$ is T_3-tri open set and T_4-tri open set but it is not open in any of the three topological spaces. Also, $\{3\}$ is T_3-tri closed set and T_4-tri closed set, but it is not closed in any of the three topological spaces.

The relationship between four types of tri open (closed) sets will be explained in the following theorem.

Theorem 3.3: Let $(X, \tau_1, \tau_2, \tau_3)$ be a tri topological space.

1. If A is T_1-tri open (closed) set, then A is T_3-tri open (closed), also T_4-tri open (closed) set.

2. If A is T_2-tri open (closed) set, then A is T_3-tri open (closed) also T_4-tri open (closed) set.

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3. If A is T_3-tri open (closed) set, then A is T_4-tri open (closed) set.

4. There is no relation between T_1-tri open (closed) and T_2-tri open (closed) sets.

Proof: 1. Let A is T_1-tri open set, then $A \in \tau_1 \cap \tau_2 \cap \tau_3$

$\Rightarrow A \in \tau_i$ for $i = 1,2,3$

$\Rightarrow A \in \tau_1 \cup \tau_2 \cup \tau_3$, then A is T_3-tri open set. As well $\tau_1 \cup \tau_2 \cup \tau_3 \subseteq \tau_1 \vee \tau_2 \vee \tau_3$. Therefore A is T_4-tri open set.

In case closed sets, we get a direct proof using the complement of the set and the definition of T_1-tri closed set.

2. Let A is T_2-tri open set, which means that A is open in only one of τ_i, assume $A \in \tau_1$, then

$A \in \tau_1 \cup \tau_2 \cup \tau_3 \subseteq \tau_1 \vee \tau_2 \vee \tau_3$ .Therefore A is T_3-tri open and T_4-tri open set.

In case closed sets, produced directly from the complement of the set and the definition of T_2-tri closed set

3. Result directly from the definition of T_3-tri open (closed) and T_4-tri open (closed) set.

4. In example 3.1, the set $\{1\}$ is T_1-tri open but not T_2-tri open set. Also, the set $\{2\}$ is T_2-tri open but not T_1-tri open set.

The previous relationship between four types of open and closed sets can be expressed through the following two diagrams

$T_1 - tri\ open \searrow$

T_3-tri open$\Longrightarrow$ T_4-tri open.

$T_2 - tri\ open \nearrow$

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$T_1 \ -tri\ closed \searrow$

T_3-tri closed$\Longrightarrow$ T_4-tri closed.

$T_2 - tri\ closed \nearrow$

……………………………………………………….

$\mathrm{T}_1 - \mathrm{tri\ open(closed)} \nLeftrightarrow \mathrm{T}_2 - \mathrm{tri\ open(closed)}.$

Result 3.1: The converse of the above theorem is not generally true as shown in the following example

Example 3.3: Let $X = \{a,b,c\}, \tau_1 = \{X,\varphi,\{a\}\}, \tau_2 = \{X,\varphi,\{b,c\}\}, \tau_3 = \{X,\varphi,\{a\},\{a,b\},\{a,c\}\}$.

The set of all T_1-tri -open sets is $\{X,\varphi\}$, the set of all T_1-tri -closed sets is $\{X,\varphi\}$.

The set of all T_2-tri -open sets is $\{\{a,b\},\{a,c\},\{b,c\}\}$.

The set of all T_2-tri -closed sets is $\{\{c\},\{b\},\{a\}\}$.

The set of all T_3-tri -open sets is $\{X,\varphi,\{a\},\{b,c\},\{a,b\},\{a,c\}\}$.

The set of all T_3-tri -closed sets is $\{\varphi,X,\{b,c\},\{a\},\{c\},\{b\}\}$.

The set of all T_4-tri -open sets is $\{X,\varphi,\{a\},\{b,c\},\{a,b\},\{a,c\},\{c\}\}$.

The set of all T_4-tri -closed sets is $\{X,\varphi,\{b,c\},\{a\},\{c\},\{b\},\{a,b\}\}$.

Note that:

$\{a\},\{b,c\},\{a,b\},\{a,c\}\ are\ T_3$-tri open sets but not T_1-tri open. $X,\varphi,\{a\}\ are\ T_3$-tri open sets but not T_2-tri open.

$\{b,c\},\{a\},\{c\},\{b\}\ are\ T_3$-tri closed sets but not T_1-tri closed. $\varphi,X,\{b,c\}\ are\ T_3$-tri closed set but not T_2-tri closed.

$\{c\} is\ T_4$-tri open sets but not T_3-tri open. $\{a,b\}\ is\ T_4$-tri closed sets but not T_3-tri closed sets.

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4. T_i −Interior and T_i −Closure Operators in Tri-Topological Spaces:

Based on new types of open sets, we shall introduce concepts of T_i − interior and T_i −closure operators for any nonempty finite set in tri- topological spaces and their properties.

Definition 4.1: Let $(X, \tau_1, \tau_2, \tau_3)$ be a tri topological space, $A \subseteq X$. An element $x \in A$ is called an interior point of A from type T_i , if there exists a T_i-tri open set called V , such that $x \in V \subseteq A$.

The set of all interior points of A from type T_i is called the interior of A from type T_i and is denoted by $T_i - int(A)$.

Theorem 4.1: Let $(X, \tau_1, \tau_2, \tau_3)$ be a tri topological space. $A \subseteq X$. Then:

$T_i - int(A)$= union of all T_i-tri open sets contained in A.

$T_i - int(A) = \bigcup\{V; V \subseteq A, V is\ T_i - tri\ open\ set\}$.

Proof: Let $x \in T_i - int(A)$ then there exist a T_i-tri open set called V , such that $x \in V \subseteq A$

So $x \in \bigcup\{V; V \subseteq A, V is\ T_i - tri\ open\ set\}$. Therefore $T_i - int(A) \subseteq \bigcup\{V; V \subseteq A, V is\ T_i - tri\ open\ set\}$.

Now, for the opposite direction let $x \in \bigcup\{V; V \subseteq A, V is\ T_i - tri\ open\ set\}$

Then $x \in V_0 \subseteq A, V_0 is\ a\ T_i - tri\ open\ set$. So $x \in T_i - int(A)$.

Therefore $T_i - int(A) = \bigcup\{V; V \subseteq A, V is\ T_i - tri\ open\ set\}$.

Definition 4.2: Let $(X, \tau_1, \tau_2, \tau_3)$ be a tri topological space. $A \subseteq X$. The intersection of all T_i-tri closed sets containing A is called the tri-closure of A from type T_i and is denoted by $T_i - cl(A)$.

Theorem 4.2: Let $(X, \tau_1, \tau_2, \tau_3)$ be a tri topological space. $A \subseteq X$. Then:

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$T_i - cl(A) = \cap\{C; A \subseteq C, Cis\ T_i - tri\ closed\ set\}$.

Proof: it is obvious

Theorem 4.3: Let $(X, \tau_1, \tau_2, \tau_3)$ be a tri topological space. $A \subseteq X$. Then we have:

1. $T_i - int(X) =$ $T_i - cl(X) = X$ and $T_i - int(\emptyset) = T_i - cl(\emptyset) = \emptyset$.
2. $T_i - int(A) \subseteq A \subseteq T_i - cl(A)$.
3. $T_i - int(A)$ is T_i-tri open set and $T_i - cl(A)$ is T_i-tri closed set.
4. $T_i - int(A)$ is the largest T_i-tri open sets contained in A.
5. $T_i - cl(A)$ is the smallest T_i-tri closed set containing A .

Proof: 1, 2. They are clear from definitions.

3. Since $T_i - int(A) = \cup\{V; V \subseteq A, Vis\ T_i - tri\ open\ set\}$ and since union of infinite number of T_i-tri open sets is T_i-tri open set. Also for $T_i - cl(A)$ is T_i-tri closed set.

4, 5. From definitions directly.

Example 4.1: Let $X = \{3,4,5,6\}$, $\tau_1 = \{X, \varphi, \{3\}, \{4,5,6\}\}$, $\tau_2 = \{X, \varphi, \{4,5,6\}\}$,

$\tau_3 = \{X, \varphi, \{6\}, \{4,5,6\}\}$. So $(X, \tau_1, \tau_2, \tau_3)$ be a tri topological space. Let $A = \{4,5,6\} \subseteq X$.

$\tau_{I=}\{X, \varphi, \{4,5,6\}\}$, $\sigma_O = \{X, \varphi, \{3\}, \{6\}\}$.

$\sigma_\cup = \{X, \varphi, \{3\}, \{4,5,6\}, \{6\}\}$. $\tau_S = \{X, \varphi, \{3\}, \{4,5,6\}, \{6\}, \{3,6\}\}$.

Therefore, we have:

$T_1 - int(A) = \{4,5,6\}, T_1 - cl(A) = X$.

$T_2 - int(A) = \varphi, T_2 - cl(A) = \{4,5,6\}$.

$T_3 - int(A) = \{4,5,6\}, T_3 - cl(A) = \{4,5,6\}$.

$T_4 - int(A) = \{4,5,6\}, T_4 - cl(A) = \{4,5,6\}$.

Theorem 4.4: Let $(X, \tau_1, \tau_2, \tau_3)$ be a tri topological space. $A \subseteq X$. Then the following hold:

1. A is T_i-tri open iff $T_i - int(A) = A$.
2. A is T_i-tri closed iff $T_i - cl(A) = A$.

Proof: 1. Let A is T_i-tri open, so $A \in \cup\{V; V \subseteq A, V is\ T_i - tri\ open\ set\} \subseteq A$.Therefore

$T_i - int(A) = A$. For the reverse, since $T_i - int(A) = A$ then A is T_i-tri open.

2. It is similar to a number1.

Theorem 4.5: Let $(X, \tau_1, \tau_2, \tau_3)$ be a tri topological space. $A, B \subseteq X$. Then the following properties hold:

1. If $A \subseteq B$ then $T_i - int(A) \subseteq T_i - int(B)$.
2. $T_i - int(A) \cup T_i - int(B) \subseteq T_i - int(A \cup B)$.
3. $T_i - int(A \cap B) \subseteq T_i - int(A) \cap T_i - int(B)$.
4. $T_i - int(T_i - int(A)) = T_i - int(A)$.

Proof: By using theorem 4.1and theorem 4.2, we get the proof, directly.

Theorem 4.6: Let $(X, \tau_1, \tau_2, \tau_3)$ be a tri topological space. $A, B \subseteq X$. Then the following properties hold:

1. If $A \subseteq B$ then $T_i - cl(A) \subseteq T_i - cl(B)$.
2. $T_i - cl(A \cup B) = T_i - cl(A) \cup T_i - cl(B)$.
3. $T_i - cl(A \cap B) \subseteq T_i - cl(A) \cap T_i - cl(B)$.
4. $T_i - cl(T_i - cl(A)) = T_i - cl(A)$.

Proof: It is similar to the proof of theorem 4.5 and considering the definition of $T_i - cl(A)$.

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Definition 4.3: Let $(X, \tau_1, \tau_2, \tau_3)$ be a tri topological space, for i=1,2,3,4. A is T_i-tri open. Then we have:

$int_{\tau_1}(int_{\tau_2}(int_{\tau_3} A))$ = union of all T_i-tri open sets contained in A and
$cl_{\tau_1}(cl_{\tau_2}(cl_{\tau_3} A))$ = Intersection of all T_i-tri closed sets containing A.

Theorem 4.7: Let $(X, \tau_1, \tau_2, \tau_3)$ be a tri topological space.then

1. A is T_1-tri open iff $A \subset int_{\tau_1}(int_{\tau_2}(int_{\tau_3}$ A)).
2. A is T_1-tri closed iff $A \supset cl_{\tau_1}(cl_{\tau_2}(cl_{\tau_3}(A))$.

Proof: 1. Let A is T_1-tri open, then A is open with respect to each topology.
Hence A = int_{τ_i} A for i = 1, 2, 3.
$int_{\tau_1}(int_{\tau_2}(int_{\tau_3} A)) = int_{\tau_1}(int_{\tau_2} A) = int_{\tau_1} A = A$
Hence $A \subset int_{\tau_1}(int_{\tau_2}(int_{\tau_3} A))$. Conversely, suppose we have $A \subset int_{\tau_1}(int_{\tau_2}(int_{\tau_3} A))$, so $int_{\tau_1}(int_{\tau_2}(int_{\tau_3} A)) \subset int_{\tau_1}(int_{\tau_2}$ A) $\subset int_{\tau_1} A \subset A$. Then we have $A = int_{\tau_1}(int_{\tau_2}(int_{\tau_3} A))$, which implies $A = int_{\tau_i}(A)$ for i = 1, 2, 3 and therefore A isT_1-tri open.
2. Let A is T_1-tri closed $\Rightarrow A^C$ is T_1-tri open.

$$\Rightarrow A^C \subset int_{\tau_1}(int_{\tau_2}(int_{\tau_3} A^C))$$
$$\Rightarrow A^C \subset int_{\tau_1}(int_{\tau_2}(cl_{\tau_3}(A))^C$$
$$\Rightarrow A^C \subset int_{\tau_1}(cl_{\tau_2}(cl_{\tau_3}(A))^C$$
$$\Rightarrow A^C \subset [cl_{\tau_1}(cl_{\tau_2}(cl_{\tau_3}(A)))]^C$$
$$\Rightarrow A \supset cl_{\tau_1}(cl_{\tau_2}(cl_{\tau_3}(A)).$$

Theorem 4.8: Let $(X, \tau_1, \tau_2, \tau_3)$ be a tri topological space.
Let $A \subseteq X$.
1. A is T_3-tri open $\Rightarrow int_{\tau_1}(int_{\tau_2}(int_{\tau_3} A)). = A$.
2. A is T_3-tri closed $\Rightarrow A = cl_{\tau_1}(cl_{\tau_2}(cl_{\tau_3}(A))$.

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Proof: 1. int_{τ_1} (int_{τ_2} (int_{τ_3} A)) = Union of all T_3-tri open sets contained in A. Since A is T_3-tri open, union of all T_3-tri open sets contained in A is A. Hence int_{τ_1} (int_{τ_2} (int_{τ_3} A)) = A.
2. The proof is similar to 1.

Result 4.1: Let $(X, \tau_1, \tau_2, \tau_3)$ be a tri topological space. A is T_3-tri open. Then

$$[int_{\tau_1} (int_{\tau_2} (int_{\tau_3} A))]^C = cl_{\tau_1} (cl_{\tau_2} (cl_{\tau_3} (A)^C)).$$

Remark 4.1: The subsequent example explains that the inverse of the above theorem does not true in general case.
Example 4.2: In Example 2.1, let $A = \{a, b\} = \{a\} \cup \{b\}$, $\{a\}$ and $\{b\}$ are T_3-tri open sets contained in A. Hence A $= int_{\tau_1}\left(int_{\tau_2}\left(int_{\tau_3} A\right)\right)$. However, $A = \{a, b\}$ is not T_3-tri open.
Theorem 4.9: Let $(X, \tau_1, \tau_2, \tau_3)$ be a tri topological space. A is T_3-tri open. Then
1. int_{τ_1} (int_{τ_2} (int_{τ_3} A)) = union of all T_3-tri open sets contained in$A$.
2.cl_{τ_1} (cl_{τ_2} (cl_{τ_3} A)) = Intersection of all T_3-tri closed sets containing$A$.
Proof: 1. int_{τ_1} (int_{τ_2} (int_{τ_3} A)) = Union of all T_3-tri open sets contained in A. Since A is T_3-tri open, union of all T_3-tri open sets contained in A is A. Hence int_{τ_1} (int_{τ_2} (int_{τ_3} A)) = A.
2. The proof is similar to 1.

5. Operations on T_i-tri Open sets:

We will perform intersection and union operations on different types of tri open sets and whether they preserve their type or not.
Theorem 5.1: Let $(X, \tau_1, \tau_2, \tau_3)$ be a tri topological space. We have:
1. Arbitrary union of T_1-tri open sets is T_1- tri open.

2. Arbitrary intersection of T_1- tri closed sets is T_1- tri closed.

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Proof: 1. Let $\{A_i, i \in I\}$ be a family of T_1-tri open sets in X. By theorem 2.11, for each $i \in I$, A_i is T_1-tri open iff $A_i \subset int_{\tau_1}(int_{\tau_2}(int_{\tau_3} A_i))$.

Hence $\cup A_i \subset \cup [int_{\tau_1}(int_{\tau_2}(int_{\tau_3} A_i))]$.

$\subset int_{\tau_1} \cup [int_{\tau_2}(int_{\tau_3} A_\alpha))]$.

$\subset int_{\tau_1} \; int_{\tau_2} \cup [(int_{\tau_3} A_\alpha))]$.

$\subset int_{\tau_1} int_{\tau_2} \; int_{\tau_3}[\cup A_\alpha]$. Therefore, $\cup A_i$ is T_1-tri open.

2. Let $\{B_i, i \in I\}$ be a family of T_1-tri closed sets in X. Let $A_i = {B_i}^C$, so $\{A_i, i \in I\}$ is a family of T_1-tri open sets in X and since arbitrary union of T_1-tri open sets is T_1-tri open. Hence $\cup A_i$ is T_1-tri open and so $(\cup A_i)^C$ is T_1-tri closed. i.e., $\cap {A_i}^C$ is T_1-tri closed (i.e.) $\cap B_i$ is T_1-tri closed. Hence, arbitrary intersection of T_1-tri closed sets is T_1-tri closed.

Theorem 5.2: Let $(X, \tau_1, \tau_2, \tau_3)$ be a tri topological space.

1. The arbitrary union of T_4-tri open sets is T_4-tri open set.
2. The finite intersection of T_4-tri closed sets is T_4-tri closed set.

Proof: 1-Let $\{A_i, i \in I\}$ be a family of T_3-tri open sets, so $A_i \in \tau_1 \vee \tau_2 \vee \tau_3$ for all $i \in I$, but $(X, \tau_1 \vee \tau_2 \vee \tau_3)$ is a supremum topological space, we have the arbitrary union of its sets is a set to which it belongs, then we have $\cup A_i \in \tau_1 \vee \tau_2 \vee \tau_3$ for all $i \in I$, therefore $\cup A_i$ is T_4-tri open set.

2- Let $\{A_i, i = 1, \dots n\}$ are finite family of T_4-tri closed sets, then we have $(A_i)^C$ is T_4-tri open set for all $i = 1, \dots n$ and $(A_i)^C \in \tau_1 \vee \tau_2 \vee \tau_3$ but $(X, \tau_1 \vee \tau_2 \vee \tau_3)$ is supremum topological space then $\cup ((A_i)^C \in \tau_1 \vee \tau_2 \vee \tau_3$, for all $i \in I$. Therefore $\cup ((A_i)^C)$ is T_4-tri open set and by definition 2.2, we have $(\cup (A_i)^C)^C = \cap ((A_i)^C)^C = \cap A_i$ sets is T_4-tri closed set

Result 5.1: In a tri topological space $(X, \tau_1, \tau_2, \tau_3)$

1. Union of two T_3-tri open (closed) sets need not be T_3-tri open (closed).
2. Intersection of two T_3-tri open (closed) sets need not be T_3-tri open (closed).

Proof: since T_3-tri open (closed) sets belong to $\sigma_{\cup}$ and $\sigma_{\cup}$ is not topology space. Therefore, union and intersection not achieved.

Example 5.1:1.In Example 2.1, $\{a\},\{b\}$ are T_3-tri open but $\{a\}\cup\{b\}=\{a,b\}$is not be T_3-tri open.

2. Let$X=\{1,2,3\}$, $\tau_1=\{X,\varphi,\{1\}\}, \tau_2=\{X,\varphi,\{2\},\{2,3\}\}$, $\tau_3=\{X,\varphi,\{2\},\{1,3\}\}$.

T_3-tri open sets are $X,\varphi,\{1\},\{2\},\{2,3\},\{1,3\}$.$T_3$-tri closed sets are $\varphi,X,\{2,3\},\{1,3\},\{1\},\{2\}$. we have $\{1\}\cup\{2\}=\{1,2\}$ is not T_3-tri open (closed) set. And $\{2,3\}\cap\{1,3\}=\{3\}$ is not T_3-tri open (closed) set.

Result 5.2: In a tri topological space (X,τ_1,τ_2,τ_3)

1. Union of two T_2-tri open (closed) sets need not be T_2-tri open (closed).
2. Intersection of two T_2-tri open (closed) sets need not be T_2-tri open (closed).

Proof: since T_2-tri open (closed) sets belong to σ_O and σ_O is not topology space. Therefore, union and intersection not achieved.

Example 5.2: In Example 2.3, $\{2\},\{4\}$ are T_2-tri open but $\{2\}\cup\{4\}=\{2,4\}$is not be T_2-tri open.

$\{1,2\},\{1,3\}$ are T_2-tri open but $\{1,2\}\cap\{1,3\}=\{1\}$ is not T_2-tri open

$\{3,4\},\{2,4\}$ are T_2-tri closed but $\{3,4\}\cap\{2,4\}=\{4\}$ is not T_2-tri closed set. Also, $\{3,4\}\cup\{2,4\}=\{2,3,4\}$ is not T_2-tri closed set.

6. T_i-rough sets:

The main idea of Pawlak's work [1] depends on a set of objects X called the universe and E is an equivalence relation, representing our knowledge about the elements of X. To characterize any vague concept $A\subseteq X$, with respect to E, let $x\in U$. An equivalence class of an element$x$, determined by$E$, is $[x]_E=\{y\in X: E(x)=E(y)\}$. vagueness is expressed through a pair of precise concepts called lower and upper approximations of a set A, which defined as

$\underline{E}(A) = \cup \{[x]E : [x]E \subseteq A\}$, $\overline{E}(A) = \cup \{[x]E : [x]E \cap A = \varphi\}$. Rough set theory employs the boundary region $b\,(A)$ of a set $A \subseteq X$, where

$$b\,(A) = \overline{E}(A) - \underline{E}(A).$$

If the boundary region of a set is empty, the set is considered crisp (exact); otherwise, the set is rough (inexact). A nonempty boundary region indicates that our knowledge about the set is insufficient to define it precisely.

Topological rough approximations proposed by Wiweger [12] represent the first generalization of rough set approximations based on topological structures. He demonstrated that lower and upper approximations can be replaced by the interior and closure operators, defined as follows:

- For a given set$A \subseteq X$, the lower approximations$A$, denoted as $\underline{E}(A\ )$ is the largest open set contained within$A$. It represents the set of elements that definitely belong to$A$.
- The upper approximations of$A$, denoted as $\bar{E}(A)$ is the smallest closed set that contains A. It includes all the points in A and all its limit points.

$$\underline{E}(A) = int(A) = \bigcup\{G \in \tau : G \subseteq A\}.$$
$$\bar{E}(A) = cl(A) = \bigcap\{G \in \tau c : A \subseteq G\}.$$

In this section, we will rely on Wiweger's topological idea and apply it in tri topological spaces to arrive at four types of lower approximations and four types of upper approximations for any non-empty set $A \subseteq X$.

Definition 6.1: Let $(X, \tau_1, \tau_2, \tau_3)$ be a tri topological space. $A \subseteq X$. For the same index i, the lower, upper and boundary approximations of A are defined respectively as:

$$T_i - \underline{E}(A) = T_i - int(A) = \bigcup\{V; V \subseteq A, V is\ T_i - tri\ open\ set\}$$

$$T_i - \bar{E}(A) = T_i - cl(A) = \bigcap\{C; A \subseteq C, C is\ T_i - tri\ closed\ set\}.$$

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$$T_i - b(A) = T_i - \bar{E}(A) - T_i - \underline{E}(A).$$

Example 6.1: In example 4.1:

$T_1 - \underline{E}(A) = \{4,5,6\}, T_1 - \bar{E}(A) = X, T_1 - b(A) = \{3\}.$
$T_2 - \underline{E}(A) = \varphi, T_2 - \bar{E}(A) = \{4,5,6\}. T_2 - b(A) = \{4,5,6\}$
$T_3 - \underline{E}(A) = \{4,5,6\}, T_3 - \bar{E}(A) = \{4,5,6\}. T_3 - b(A) = \varphi.$
$T_4 - \underline{E}(A) = \{4,5,6\}, T_4 - \bar{E}(A) = \{4,5,6\}. T_4 - b(A) = \varphi.$

Theorem 6.1: Let $(X, \tau_1, \tau_2, \tau_3)$ be a tri topological space. $A \subseteq X$.

1. $T_1 - \underline{E}(A) \subseteq T_3 - \underline{E}(A) \subseteq T_4 - \underline{E}(A).$
2. $T_2 - \underline{E}(A) \subseteq T_3 - \underline{E}(A) \subseteq T_4 - \underline{E}(A).$
3. $T_4 - \bar{E}(A) \subseteq T_3 - \bar{E}(A) \subseteq T_1 - \bar{E}(A).$
4. $T_4 - \bar{E}(A) \subseteq T_3 - \bar{E}(A) \subseteq T_2 - \bar{E}(A).$
5. There is no relation between $T_1 - \underline{E}(A)$ and $T_2 - \underline{E}(A)$. Also, $T_1 - \bar{E}(A)$ *and* $T_2 - \bar{E}(A)$.

Proof: 1. Since every set belongs to τ_I is open set in σ_U and all sets belong to σ_U is open in τ_S then $T_1 - \underline{E}(A) \subseteq T_3 - \underline{E}(A) \subseteq T_4 - \underline{E}(A)$.

2. It is similar to 1.

3. Applying the complement of the previous relation and definition of the lower and upper approximations of A, we get what is required.

4. It is similar to 3.

5. As is clear in the example 6.1.

Definition 6.2: Let $(X, \tau_1, \tau_2, \tau_3)$ be a tri topological space. Let $A \subseteq X$. We determine the degree of crispness of A by the accuracy measure:

$$T_i - \alpha_E(A) = \left| \frac{T_i - \underline{E}(A)}{T_i - \bar{E}(A)} \right|, \text{ where } T_i - \bar{E}(A) \neq \varphi.$$

Definition 6.3: Let $(X, \tau_1, \tau_2, \tau_3)$ be a tri topological space. Let $A \subseteq X$. For any element $x \in X$, rough membership relations to A are defined as $x \underline{\in}_{T_i} A$ if $x \in T_i - \underline{E}(A)$ and $x \overline{\in}_{T_i} A$ where $x \in T_i - \bar{E}(A)$.

Proposition 6.1: Let $(X, \tau_1, \tau_2, \tau_3)$ be a tri topological space. $A \subseteq X$. Then

$$T_i - \underline{E}(X) = T_i - \bar{E}(X) = X,$$
$$T_i - \underline{E}(\varphi) = T_i - \bar{E}(\varphi) = \varphi$$

Proof: it is obvious.

Theorem 6.2: Let $(X, \tau_1, \tau_2, \tau_3)$ be a tri topological space. $A, B \subseteq X$. Then the following properties hold for the same index i:

1. $T_i - \underline{E}(A) \subseteq A \subseteq T_i - \bar{E}(A)$.
2. If $A \subseteq B$ then $T_i - \underline{E}(A) \subseteq T_i - \underline{E}(B)$and $T_i - \bar{E}(A) \subseteq T_i - \bar{E}(B)$.
3. $(T_i - \bar{E}(A)) \cup (T_i - \bar{E}(B)) \subseteq T_i - \bar{E}(A \cup B)$.
4. $T_i - \underline{E}(A \cap B) \subseteq (T_i - \underline{E}(A)) \cap (T_i - \underline{E}(B))$.
5. $T_i - \underline{E}(A \cup B) = (T_i - \underline{E}(A)) \cup (T_i - \underline{E}(B))$.
6. $T_i - \bar{E}(A \cap B) \subseteq T_i - \bar{E}(A) \cap T_i - \bar{E}(B)$.
7. $T_i - \underline{E}\left(T_i - \underline{E}(A)\right) = T_i - \underline{E}(A)$ and $T_i - \bar{E}\left(T_i - \bar{E}(A)\right) = T_i - \bar{E}(A)$.
8. $T_i - \underline{E}(A^C) = (T_i - \bar{E}(A))^C$and $T_i - \bar{E}(A^C) = (T_i - \underline{E}(A))^C$.

Proof: By using the properties of T_i −interior and T_i −closure operators, also from definitions 3.3 and 3.4, we get the proof, directly.

Definition 6.4: Let $(X, \tau_1, \tau_2, \tau_3)$ be a tri topological space. $A \subseteq X$. For the same index i, we have:

1. A is called a T_i-tri rough set if $T_i - \underline{E}(A) \neq T_i - \bar{E}(A)$.
2. A is called a T_i-tri exact set if $T_i - \underline{E}(A) = T_i - \bar{E}(A)$.

Theorem 6.3: For a set $A \subseteq X$ in tri topological space, for the same index i

A is a $T_i -$ rough set iff $T_i - b(A) \neq \varphi$,

A is a $T_i -$ exact set iff $T_i - b(A) = \varphi$

Proof: it is obvious.

Definition 6.5: Let $(X, \tau_1, \tau_2, \tau_3)$ be a tri topological space. Let $A \subseteq X$. For the same index i, we have:

1. A is called a tri rough set of Type 1 in X if A is rough set in τ_I. We denoted to it as (T_1-tri rough set).
2. A is called a tri rough set of Type 2 in X if A is rough in σ_O. We denoted to it as (T_2-tri rough set).
3. A is called a tri rough set of Type 3 in X if A is rough in σ_U. We denoted to it as (T_3-tri rough set).
4. A is called a tri rough set of Type 4 in X if A is rough in τ_S. We denoted to it as (T_4-tri rough set).

Example 6.2: In example 6.1: A is T_1-tri rough and T_2-tri rough set. Also, A is T_3-tri exact and T_4-tri exact set.

Theorem 6.4: In a tri topological space$(X, \tau_1, \tau_2, \tau_3)$. $A \subseteq X$, the following properties are satisfied

1. A is T_1 $-$exact $\Longrightarrow$ A is T_3 $-$exact$\Longrightarrow$ A is T_4 $-$exact.
2. A is T_2 $-$exact $\Longrightarrow$ A is T_3 $-$exact$\Longrightarrow$ A is T_4 $-$exact.
3. A is T_4 $-$rough $\Longrightarrow$ A is T_3 $-$rough$\Longrightarrow$ A is T_1 $-$rough.
4. A is T_4 $-$rough $\Longrightarrow$ A is T_3 $-$rough$\Longrightarrow$ A is T_2 $-$rough.

Proof: 1. Let A is T_1 $-$exact, this means that $T_1 - \underline{E}(A) = T_1 - \bar{E}(A)$…. (*). Also,

$T_1 - \underline{E}(A) \subseteq T_3 - \underline{E}(A) \subseteq T_4 - \underline{E}(A)$ and from theorem 6.2, we have $T_4 - \underline{E}(A) \subseteq T_4 - \bar{E}(A)$**.** Now by theorem 6.1, we have $T_1 - \underline{E}(A) \subseteq T_3 - \underline{E}(A) \subseteq T_4 - \underline{E}(A) \subseteq T_4 - \bar{E}(A) \subseteq T_3 - \bar{E}(A) \subseteq T_1 - \bar{E}(A)$ ... $(**)$

Hence, from (*) and $(**)$ we have A is T_3 $-$exact and T_4 $-$exact.

2. It is similar to 1.

3. Let A is T_4 $-$rough, this means that $T_4 - \underline{E}(A) \neq T_4 - \bar{E}(A)$. Looking at the relation$(**)$, then $T_3 - \underline{E}(A) \neq T_3 -$

$\bar{E}(A)$. Also, $T_1 - \underline{E}(A) \neq T_1 - \bar{E}(A)$. Therefore A is T_3 −rough and T_1 −rough.

4. It is similar to 3.

Definition 6.6: In a tri topological space$(X, \tau_1, \tau_2, \tau_3)$, for a subset $A \subseteq X$ is called:

1. T_i −Totally definable (T_i −exact), if $T_i - \underline{E}(A) = T_i - \bar{E}(A) = A$.
2. T_i −Internally definable, if $T_i - \underline{E}(A) = A$ and $T_i - \bar{E}(A) \neq A$.
3. T_i −Externally definable, if $T_i - \underline{E}(A) \neq A$ and $T_i - \bar{E}(A) = A$.
4. T_i −Rough, if $T_i - \underline{E}(A) \neq A$ and $T_i - \bar{E}(A) \neq A$.

Example 6.3: In example 6.1:

A is T_3 *and* T_4- totally definable (T_3 *and* T_4- exact).

A is T_1 −iinternally definable.

A is T_2 − externally definable.

CONCLUSION:

The types of open sets are used in many real-life applications, so it was necessary to work on developing them in various spaces to suit different life situations which opened the horizon for us to define new types of rough sets using new different types of lower and upper approximations for any set in tri topological spaces and to study the relationship between them.

The most important results we reached are as follows:

- We introduced new types of open (closed) sets called T_i-tri open (closed) sets, $i = 1,2,3,4$.
- The relationships between these sets is:

$$T_1 - tri\ open \Longrightarrow T_3\text{-tri open} \Longrightarrow T_4\text{-tri open.}$$

$$T_1 - tri\ closed \Longrightarrow T_3\text{-tri closed} \Longrightarrow T_4\text{-tri closed.}$$

$$\mathrm{T_1 - tri\ open(closed)} \nLeftrightarrow \mathrm{T_2 - tri\ open(closed).}$$

- We defined concepts of T_i− interior and T_i −closure operators in tri- topological spaces and their properties.

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- For one set, we defined four different types of lower and four different types of upper approximations.
- The relationships between these operators is:

$$T_1 - \underline{E}(A) \subseteq T_3 - \underline{E}(A) \subseteq T_4 - \underline{E}(A).$$

$$T_2 - \underline{E}(A) \subseteq T_3 - \underline{E}(A) \subseteq T_4 - \underline{E}(A).$$

$$T_4 - \bar{E}(A) \subseteq T_3 - \bar{E}(A) \subseteq T_1 - \bar{E}(A).$$

$$T_4 - \bar{E}(A) \subseteq T_3 - \bar{E}(A) \subseteq T_2 - \bar{E}(A).$$

- We introduced new types of rough sets called $T_i -$ rough set , $i = 1,2,3,4$.
- We have found that the same set can be either rough or defined depending on the standards by which we measure it, which helps to make more accurate judgments in different complex situations.
- The relationships between $T_i -$ rough set is:

$$\mathrm{T}_1 - \text{exact} \Rightarrow \mathrm{T}_3 - \text{exact} \Rightarrow \mathrm{T}_4 - \text{exact}.$$

$$\mathrm{T}_2 - \text{exact} \Rightarrow \mathrm{T}_3 - \text{exact} \Rightarrow \mathrm{T}_4 - \text{exact}.$$

$$\mathrm{T}_4 - \text{rough} \Rightarrow \mathrm{T}_3 - \text{rough} \Rightarrow \mathrm{T}_1 - \text{rough}.$$

$$\mathrm{T}_4 - \text{rough} \Rightarrow \mathrm{T}_3 - \text{rough} \Rightarrow \mathrm{T}_2 - \text{rough}.$$

- We hope that these results are just the beginning of applying the new types of openness and rough sets to various topological topics.

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